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Group graded algebras and multiplicities bounded by a constant

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ABSTRACT

Let G be a finite group and A a G-graded algebra over a field of characteristic zero. When A is a PI-algebra, the graded codimensions of A are exponentially bounded and one can study the corresponding graded cocharacters via the representation theory of products of symmetric groups. Here we characterize in two different ways when the corresponding multiplicities are bounded by a constant.

1. Introduction

Let F be a field of characteristic zero and A an F-algebra graded by a finite group G. The graded polynomial identities of A have been extensively studied in the past years (see for instance [3,4,7,12]). In particular in [1,2,9], inspired by Giambruno and Zaicev [11], the authors proved the existence of the G-exponent for any graded algebra satisfying an ordinary polynomial identity (PI-algebra). Motivated by these results here we study a specific property of the graded polynomial identities satisfied by A.

When A is a PI-algebra the sequence of graded codimensions of A is exponentially bounded (see [10]) and one can successfully apply the representation theory of the symmetric group in order to study the graded identities of A. More precisely, one considers, for every $n_1, \ldots, n_s \geq 0$, the spaces $P_{n_1,\ldots,n_s}$ of multilinear graded polynomials in variables of fixed homogeneous degree and acts on it with the direct product of symmetric groups $S_{n_1} \times \cdots \times S_{n_s}$. The space $P_{n_1,\ldots,n_s}$ modulo $Id^G(A)$, the ideal of graded identities of A, becomes an $S_{n_1} \times \cdots \times S_{n_s}$-module and $\chi^G_{n_1,\ldots,n_s}(A)$ is its character. By complete reducibility we write such character as a sum of irreducible characters with corresponding multiplicities.

The purpose of this paper is to characterize the ideal of graded identities of A in case the multiplicities are bounded by a constant. We shall do this in three different ways. In fact we shall prove that the multiplicities are bounded by a constant if and only if $Id^G(A) \not\supseteq Id^G(UT^G_2)$, where UT^G_2 is the algebra of upper triangular matrices of order two with any possible G-grading.

Another characterization will be given in terms of S_n -characters: in fact we shall prove that the characters appearing with non-zero multiplicity in $\chi^G_{n_1,\ldots,n_s}(A)$ have corresponding Young diagrams contained in a special hook shaped part of the plane.

We point out that when G is an abelian group, by using the well-known duality between G-gradings and G-actions, one can translate these results in the language of $G \wr S_n$- characters. In this setting one considers the space P^G_n of multilinear G-graded polynomials in n variables and acts on it with the wreath product $G \wr S_n$. The character $\chi^G_n(A)$ of the corresponding quotient modulo $P^G_n(A) = \frac{P^G_n}{P^G_n \cap Id^G(A)}$ is called the $G \wr S_n$- cocharacter of A. Since the multiplicities in $\chi^G_n(A)$ and $\chi^G_{n_1,\ldots,n_s}(A)$ are the same (see [12]), the above result can be read in this setting.

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Finally, we remark that these results are inspired by and generalize a theorem of Mishchenko et al. [14] concerning the ordinary (non graded) case.

2. Preliminaries

Throughout the paper F will denote a field of characteristic zero and A an associative F-algebra satisfying a non-trivial polynomial identity (PI-algebra). Let $F\langle X\rangle$ be the free associative algebra on a countable set $X = \{x_1, x_2, \ldots\}$ and $Id(A) = \{f \in F\langle X\rangle \mid f \equiv 0 \text{ on } A\}$ the T-ideal of (ordinary) polynomial identities of A. For every $n \geq 1$ we denote by $P_n = \mathrm{span}_F\{x_{\sigma(1)}\cdots x_{\sigma(n)} \mid \sigma \in S_n\}$ the space of multilinear polynomials in $x_1, \ldots, x_n$ and $P_n(A) = P_n/(P_n \cap Id(A))$.

Suppose that the algebra A is graded by a finite group G. Let $G = \{g_1 = e, g_2, \ldots, g_s\}$ and let $A = \oplus_{i=1}^s A_{g_i}$ be the decomposition of A into its homogeneous components. Hence $A_{g_i}A_{g_j} \subseteq A_{g_ig_j}$, for all $i, j = 1, \ldots, s$.

We denote by $F\langle X, G\rangle$ the free associative G-graded algebra of countable rank over F. Here the set X decomposes as $X = \cup_{i=1}^s X_{g_i}$, where the sets $X_{g_i} = \{x_{1,g_i}, x_{2,g_i}, \ldots\}$ are disjoint, and the elements of X_{g_i} have homogeneous degree g_i. The algebra $F\langle X, G\rangle$ has a natural G-grading $F\langle X, G\rangle = \oplus_{g\in G}\mathcal{F}_g$, where $\mathcal{F}_g$ is the subspace of $F\langle X, G\rangle$ spanned by the monomials $x_{i_1,g_{j_1}}\cdots x_{i_t,g_{j_t}}$ of homogeneous degree $g = g_{j_1}\cdots g_{j_t}$.

Recall that an element of $F\langle X, G\rangle$ is called a graded polynomial. Also, f is a graded (polynomial) identity of the algebra A, and we write $f \equiv 0$, in case f vanishes under all graded substitutions $x_{i,g} \to a_g \in A_g$. Let $Id^G(A) = \{f \in F\langle X, G\rangle \mid f \equiv 0 \text{ on } A\}$ be the ideal of graded identities of A. Clearly, $Id^G(A)$ is invariant under all graded endomorphism of $F\langle X, G\rangle$.

Notice that if for $i \geq 1$ we set $x_i = x_{i,g_i} + \cdots + x_{i,g_s}$, then the free algebra $F\langle X\rangle$ is naturally embedded in $F\langle X, G\rangle$ and we can regard the ordinary identities of A as a special kind of graded identities.

Since $\mathrm{char}F = 0$, the multilinear polynomials of $Id^G(A)$ determine all of $Id^G(A)$. Hence for $n \geq 1$ we define

$$P_n^G = \mathrm{span}_F\{x_{\sigma(1),g_{i_{\sigma(1)}}}\cdots x_{\sigma(n),g_{i_{\sigma(n)}}} \mid \sigma \in S_n, g_{i_1}, \ldots, g_{i_n} \in G\},$$

the space of multilinear G-graded polynomials in the variables $x_{1,g_{i_1}}, \ldots, x_{n,g_{i_n}}$, $g_{i_j} \in G$. The ideal $Id^G(A)$ is determined by the sequence of subspaces $P_n^G \cap Id^G(A)$, $n = 1, 2, \ldots$, but we can consider even smaller spaces.

Let $n \geq 1$ and write $n = n_1+\cdots+n_s$ as a sum of non-negative integers. Define $P_{n_1,\ldots,n_s} \subseteq P_n^G$ to be the space of multilinear graded polynomials in which the first n_1 variables have homogeneous degree g_1, the next n_2 variables have homogeneous degree g_2, and so on. Notice that for every choice of such $n_1, \ldots, n_s$, there are $\binom{n}{n_1,\ldots,n_s}$ subspaces isomorphic to $P_{n_1,\ldots,n_s}$ where $\binom{n}{n_1,\ldots,n_s}$ denotes the multinomial coefficient. It is clear that P_n^G is the direct sum of such subspaces with $n_1 + \cdots + n_s = n$. Moreover, such decomposition is inherited by $P_n^G \cap Id^G(A)$ and we consider the spaces $P_{n_1,\ldots,n_s} \cap Id^G(A)$.

At the light of these observations, one defines

$$P_{n_1,\ldots,n_s}(A) = \frac{P_{n_1,\ldots,n_s}}{P_{n_1,\ldots,n_s} \cap Id^G(A)}.$$

The space $P_{n_1,\ldots,n_s}(A)$ is naturally endowed with a structure of $S_{n_1} \times \cdots \times S_{n_s}$-module in the following way: the group $S_{n_1} \times \cdots \times S_{n_s}$ acts on the left on $P_{n_1,\ldots,n_s}$ by permuting the variables of the same homogeneous degree; hence S_{n_1} permutes the variables of homogeneous degree g_1, S_{n_2} those of homogeneous degree g_2, etc. Since $Id^G(A)$ is invariant under this action, $P_{n_1,\ldots,n_s}(A)$ has a structure of $S_{n_1} \times \cdots \times S_{n_s}$-module and we denote by $\chi^G_{n_1,\ldots,n_s}(A)$ its character.

If $\lambda(1) \vdash n_1, \ldots, \lambda(s) \vdash n_s$, are partitions, then we write $\langle\lambda\rangle = (\lambda(1), \ldots, \lambda(s)) \vdash (n_1, \ldots, n_s)$ and we say that $\langle\lambda\rangle$ is a multipartition of $n = n_1 + \cdots + n_s$.

Since $\mathrm{char}F = 0$, by complete reducibility $\chi^G_{n_1,\ldots,n_s}(A)$ can be written as a sum of irreducible characters and we have

$$\chi^G_{n_1,\ldots,n_s}(A) = \sum_{\langle\lambda\rangle\vdash n} m_{\langle\lambda\rangle}\chi_{\lambda(1)} \otimes \cdots \otimes \chi_{\lambda(s)}, \tag{1}$$

where $m_{\langle\lambda\rangle}$ is the multiplicity of $\chi_{\lambda(1)} \otimes \cdots \otimes \chi_{\lambda(s)}$ in $\chi^G_{n_1,\ldots,n_s}(A)$. We call $\chi^G_{n_1,\ldots,n_s}(A)$ the $(n_1, \ldots, n_s)$th cocharacter of A and this will be the object of our study.

We remark that if G is an abelian group, there is a well-known duality between G-gradings and G-actions on the algebra A (one needs to assume that the base field has enough roots of 1). Through this duality one can define an action of the wreath product $G \wr S_n$ on P_n^G (see [10]). Since this action preserves $Id^G(A)$, the space $P_n^G(A) = P_n^G/(P_n^G \cap Id^G(A))$ becomes a $G \wr S_n$-module and let $\chi_n^G(A)$ be its character. The irreducible characters of $G \wr S_n$ are indexed by multipartition of n and let $\chi_{\langle\lambda\rangle}$ be the irreducible $G \wr S_n$-character associated to $\langle\lambda\rangle$. Then one writes

$$\chi_n^G(A) = \sum_{\langle\lambda\rangle\vdash n} m'_{\langle\lambda\rangle}\chi_{\langle\lambda\rangle}, \tag{2}$$

and by an obvious generalization of [8], we have that if $\langle\lambda\rangle = (\lambda(1), \ldots, \lambda(s))$ with $\lambda(1) \vdash n_1, \ldots, \lambda(s) \vdash n_s$, then in (1) and (2), $m_{\langle\lambda\rangle} = m'_{\langle\lambda\rangle}$, for all multipartitions $\langle\lambda\rangle$.

Next we state some results and reductions that we shall need throughout the paper.

Let E be the infinite dimensional Grassmann algebra generated by the elements $e_1, e_2, \ldots$ subject to the condition $e_ie_j = -e_je_i$, for all i, j. Let $E = E_0 \oplus E_1$ be the natural $\mathbb{Z}_2$-grading of E, where

$$E_0 = \mathrm{span}\{e_{i_1}\cdots e_{i_{2k}} \mid 1 \leq i_1 < \cdots < i_{2k}, k \geq 0\}$$

and

$$E_1 = \mathrm{span}\{e_{i_1} \cdots e_{i_{2k+1}} \mid 1 \le i_1 < \cdots < i_{2k+1}, k \ge 0\}.$$

Recall that if $A = A_0 \oplus A_1$ is a $\mathbb{Z}_2$-graded algebra, then the algebra $E(A) = (E_0 \otimes A_0) \oplus (E_1 \otimes A_1)$ is called the Grassmann envelope of A. Notice that if $A = \oplus_{(g,i)\in G\times\mathbb{Z}_2} A_{(g,i)}$ is a $G \times \mathbb{Z}_2$-graded algebra, we can consider the induced $\mathbb{Z}_2$-grading on A and write $A = A_0 \oplus A_1$ where $A_0 = \oplus_{g\in G} A_{(g,0)}$ and $A_1 = \oplus_{g\in G} A_{(g,1)}$. Hence in this case the Grassmann envelope of A can be regarded as a G-graded algebra via $E(A) = \oplus_{g\in G} E(A)_g$ where

$$E(A)_g = (E_0 \otimes A_{(g,0)}) \oplus (E_1 \otimes A_{(g,1)}).$$

Next we recall a very useful theorem of Aljadeff and Kanel-Belov [3], proved independently by Sviridova in [15] for abelian groups.

Theorem 2.1. *Let G be a finite group and A a G-graded PI-algebra over a field F of characteristic zero. Then there exists a field extension K of F and a finite dimensional $G \times \mathbb{Z}_2$-graded algebra B over K such that $Id^G(A) = Id^G(E(B))$.*

Another basic ingredient we shall need is the following theorem of Bahturin et al. [5] that gives a characterization of the $G \times \mathbb{Z}_2$-simple algebras.

Theorem 2.2. *Let B be a finite dimensional $G \times \mathbb{Z}_2$-graded simple algebra over an algebraically closed field F. Then B has the following structure: there exist a subgroup H of $G \times \mathbb{Z}_2$, a 2-cocycle $\alpha : H \times H \to F^*$ where the action of H on F is trivial, an integer k and a k-tuple $(a_1 = e, a_2, \ldots, a_k) \in (G \times \mathbb{Z}_2)^k$ such that B is $G \times \mathbb{Z}_2$-isomorphic to $C = F^\alpha H \otimes M_k(F)$ where for $a \in G \times \mathbb{Z}_2$, $C_a = span_F\{u_h \otimes e_{ij} \mid a = a_i^{-1} h a_j\}$. Here $u_h \in F^\alpha H$ is a representative of $h \in H$ and the e_{ij}'s are the matrix units of $M_k(F)$.*

Let UT_2 denote the algebra of 2×2 upper triangular matrices over F. By Valenti [17, Theorem 1], any G-grading on UT_2 is up to isomorphism, the elementary grading determined by (e, g), for some $g \in G$. When $g = e$, we have the trivial grading. By $var^G(UT_2^G)$ we denote the variety of G-graded algebras generated by UT_2 with some elementary G-grading. Recalling that $G = \{g_1 = e, g_2, \ldots, g_s\}$, we can state the following theorem essentially proved in [17, Theorem 3].

Theorem 2.3. *Let UT_2^G be endowed with a non-trivial G-grading determined by (e, g_i), for some $i \neq 1$. Then the T_G-ideal of graded identities of UT_2^G is generated by the polynomials $[x_{1,e}, x_{2,e}]$, $x_{1,g_i}x_{2,g_i}$ and x_{1,g_j} for all $g_j \in G$, $g_j \neq e, g_i$. Moreover, if*

$$\chi^G_{n_1,\ldots,n_s}(UT_2^G) = \sum_{\langle\lambda\rangle\vdash n} m_{\langle\lambda\rangle}\chi_{\lambda(1)} \otimes \cdots \otimes \chi_{\lambda(s)}$$

is the $(n_1, \ldots, n_s)$th cocharacter of UT_2^G, we have

(1) *$m_{\langle\lambda\rangle} = q + 1$, if $\lambda(1) = (p + q, p)$, $\lambda(i) = (1)$, and $\lambda(j) = \emptyset$, $j \neq i, 1$;*
(2) *$m_{\langle\lambda\rangle} = 1$, if $\langle\lambda\rangle = ((n), \emptyset, \ldots, \emptyset)$;*
(3) *$m_{\langle\lambda\rangle} = 0$ in all other cases.*

Proof. Let $A = UT_2^G$ be graded by the pair (e, g_i). Then $A = A_e \oplus A_{g_i}$. If we consider the canonical $\mathbb{Z}_2$-grading on UT_2, we get $A = A_0 \oplus A_1$ and $A_0 = A_e$, $A_1 = A_{g_i}$. It follows that $f(x_{1,e}, \ldots, x_{k,e}, x_{1,g_i}, \ldots, x_{l,g_i}) \in Id^G(UT_2^G)$ if and only if $f(x_{1,0}, \ldots, x_{k,0}, x_{1,1}, \ldots, x_{l,1}) \in Id^{\mathbb{Z}_2}(UT_2^{\mathbb{Z}_2})$. We then apply [17, Theorem 3] and we obtain conditions (1), (2) and (3) also for the $(n_1, \ldots, n_s)$th cocharacter of UT_2^G. $\square$

Let A be a G-graded PI-algebra over a field F. As in the ordinary case (see [12, Theorem 4.1.9]), in order to study the $(n_1, \ldots, n_s)$th cocharacter of A, we may assume that F is algebraically closed. In fact, let $\bar{F}$ be the algebraic closure of F and let $\bar{A} = A \otimes_F \bar{F}$. Then $\bar{A}$ has an induced G-grading given by

$$\bar{A}_g = A_g \otimes_F \bar{F}, \ \text{for all } g \in G,$$

and $Id^G(A) \otimes_F \bar{F} = Id^G(A \otimes_F \bar{F})$. Now, since in characteristic zero any S_n-representation is absolutely irreducible (see [13]), it follows that $\chi^G_{n_1,\ldots,n_s}(\bar{A}) = \chi^G_{n_1,\ldots,n_s}(A)$. Hence throughout the paper we shall assume that the field F is algebraically closed.

In what follows if $\mathcal{V} = var^G(A)$ is a variety of G-graded algebras generated by A, we write $\chi_{n_1,\ldots,n_s}(\mathcal{V}) = \chi_{n_1,\ldots,n_s}(A)$.

3. Structure of the Grassmann envelope

In this section, we shall study the structure of a generating G-graded algebra of a variety $\mathcal{V}$ provided $UT_2^G \notin \mathcal{V}$.

Lemma 3.1. *Let $\mathcal{V}$ be a variety of G-graded PI-algebras and suppose that $UT_2^G \notin \mathcal{V}$, for any G-grading on UT_2. Then $\mathcal{V} = var^G(E(A))$ for some finite dimensional $G \times \mathbb{Z}_2$-graded algebra A such that $A = B + J$ where $B \cong B^{(1)} \oplus \cdots \oplus B^{(m)}$ with $B^{(i)} \cong F^{\alpha_i}H_i$, $1 \le i \le m$, and $J = J(A)$. Here H_i is a subgroup of $G \times \mathbb{Z}_2$ and $\alpha_i : H_i \times H_i \to F^*$ is a 2-cocycle.*

Proof. By Theorem 2.1, we can write $\mathcal{V} = var^G(E(A))$ where $E(A)$ is the Grassmann envelope of a finite dimensional $G\times\mathbb{Z}_2$-graded algebra A.

Since the base field F is assumed to be algebraically closed, by the Wedderburn–Malcev theorem ([6]), we can write $A = B + J$ where B is a maximal semisimple subalgebra of A and $J = J(A)$ is its Jacobson radical. It is well known that J is a graded ideal. Moreover by Taft [16] we assume, as we may, that B is a $G \times \mathbb{Z}_2$-graded subalgebra of A. Hence we can write

$$B = B^{(1)} \oplus \cdots \oplus B^{(m)}$$

where every $B^{(i)}$ is a $G \times \mathbb{Z}_2$-graded simple algebra.

Now, by Theorem 2.2, for every i, $B^{(i)} \cong M_{k_i}(F) \otimes F^{\alpha_i} H_i$ for some subgroup H_i of $G \times \mathbb{Z}_2$ and 2-cocycle $\alpha_i : H_i \times H_i \to F^*$. We need to prove that for every i, $1 \le i \le m$, $k_i = 1$, i.e, $B^{(i)} \cong F^{\alpha_i} H_i$.

Suppose that for some i, $B^{(i)} \cong D = M_k(F) \otimes F^{\alpha} H$ with $k > 1$. Write $D = \oplus_{(g,a)\in G\times\mathbb{Z}_2} D_{(g,a)}$, and let $((h_1, a_1), \ldots, (h_k, a_k)) \in (G \times \mathbb{Z}_2)^k$ be the k-tuple inducing the elementary grading on $M_k(F)$. Then for any $(g, a) \in G \times \mathbb{Z}_2$, we have

$$D_{(g,a)} = \operatorname{span}\{u_{(h,b)} \otimes e_{ij} \mid (h_i, a_i)(h, b)(h_j, a_j)^{-1} = (g, a)\},$$

where $\{u_{(h,b)} \mid (h, b) \in H\}$ is the canonical basis of the twisted group algebra $F^{\alpha}H$. Note that $u_{(e,0)} \otimes e_{ii} \in D_{(e,0)}$ since $(h_i, a_i)(e, 0)(h_i, a_i)^{-1} = (e, 0)$ and $u_{(e,0)} \otimes e_{12} \in D_{(h_1h_2^{-1},a_1-a_2)}$. Hence $L \cong Fe_{11} \oplus Fe_{22} \oplus Fe_{12}$ is a subalgebra of D with induced $G \times Z_2$ grading

$$\begin{pmatrix} (e,0) & (h_1h_2^{-1}, a_1-a_2) \\ 0 & (e,0) \end{pmatrix}.$$

We write $L = L_{(e,0)} \oplus L_{(h_1h_2^{-1},a_1-a_2)}$ where $L_{(e,0)} \cong Fe_{11} + Fe_{22}$, $L_{(h_1h_2^{-1},a_1-a_2)} \cong Fe_{12}$ with induced $G \times \mathbb{Z}_2$-grading. Consider now the Grassmann envelope $E(L)$ of L. If $a_1 - a_2 = 0$, we have

$$E(L) = E_0 \otimes (L_{(e,0)} \oplus L_{(h_1h_2^{-1},0)}),$$

and if $a_1 - a_2 = 1$, we have

$$E(L) = (E_0 \otimes L_{(e,0)}) \oplus (E_1 \otimes L_{(h_1h_2^{-1},1)}).$$

Therefore, if $a_1 - a_2 = 0$, $E(L) \cong \begin{pmatrix} E_0 & E_0 \\ 0 & E_0 \end{pmatrix}$ with trivial grading. It follows that if UT_2 denotes UT_2^G with trivial grading, then $UT_2 \in var^G(E(L)) \subseteq var(E(A)) = \mathcal{V}$, a contradiction.

Suppose now that $a_1 - a_2 = 1$. Then

$$E(L)_e = (E_0 \otimes L_{(e,0)}) \oplus (E_1 \otimes L_{(e,1)}) = E_0 \otimes L_{(e,0)},$$
$$E(L)_{h_1h_2^{-1}} = (E_0 \otimes L_{(h_1h_2^{-1},0)}) \oplus (E_1 \otimes L_{(h_1h_2^{-1},1)}) = E_1 \otimes L_{(h_1h_2^{-1},1)},$$

and

$$E(L)_g = 0, \quad \text{for all } g \neq e, \quad g \neq h_1h_2^{-1}.$$

Thus $E(L) \cong \begin{pmatrix} E_0 & E_1 \\ 0 & E_0 \end{pmatrix}$ with grading $\begin{pmatrix} e & h_1h_2^{-1} \\ 0 & e \end{pmatrix}$. We will show that in this case $Id^G(E(L)) = Id^G(UT_2^G)$ where UT_2^G has grading $\begin{pmatrix} e & h_1h_2^{-1} \\ 0 & e \end{pmatrix}$.

In fact it is easy to verify that $E(L)$ satisfies the identities, $[x_{1,e}, x_{2,e}] \equiv 0$, $x_{1,g}x_{2,g} \equiv 0$, for $g = h_1h_2^{-1}$ and $x_{1,h} \equiv 0$, for all $h \neq e, g$. Thus $Id^G(UT_2^G) \subseteq Id^G(E(L))$.

On the other hand, let $f \in Id^G(E(L))$ be a multilinear polynomial and let $Q = \langle [x_{1,e}, x_{2,e}], x_{1,g}x_{2,g}, x_{1,h} \mid h \neq e, g\rangle_T$ be the T-ideal generated by the polynomials $[x_{1,e}, x_{2,e}]$, $x_{1,g}x_{2,g}$ and $x_{1,h}$. If we reduce f mod. Q, we may clearly assume that only one variable of homogeneous degree g appears in f. Hence we may assume that the polynomial f can be written in the form:

$$f = \sum_{i_1<\cdots<i_h, j_1<\cdots j_{n-h-1}} \alpha_{i_1\ldots i_h} x_{i_1,e} \cdots x_{i_h,e} x_{1,g} x_{j_1,e} \cdots x_{j_{n-h-1},e}.$$

We shall prove that for any $\{i_1, \ldots, i_h\} \subseteq \{1, \ldots, n\}$, $\alpha_{i_1\ldots i_h} = 0$. In fact if we specialize $x_{i_1,e} = \cdots = x_{i_h,e} = e_{11}$, $x_{1,g} = e_{12}$, $x_{j_1,e} = \cdots = x_{j_{n-h-1},e} = e_{22}$, f takes value $\alpha_{i_1\ldots i_h} e_{12} = 0$. This proves that $\alpha_{i_1\ldots i_h} = 0$ for all $i_1, \ldots, i_h$ and, so, $f = 0$.

We have proved that $E(L) \cong \begin{pmatrix} E_0 & E_1 \\ 0 & E_0 \end{pmatrix}$ has the same G-graded identities as UT_2^G with grading $\begin{pmatrix} e & h_1h_2^{-1} \\ 0 & e \end{pmatrix}$. It follows that $var(UT_2^G) \subseteq var(E(L)) \subseteq var(E(A))$, a contradiction. So, for all $i \geq 1$, $B^{(i)} \cong F^{\alpha}H$ and the proof is complete. □

Lemma 3.2. *Under the hypotheses of the previous lemma,*

$$\mathcal{V} = var^G(E(A_1) \oplus \cdots \oplus E(A_n))$$

where for every $i \in \{1, \ldots, n\}$, A_i is a finite dimensional $G \times \mathbb{Z}_2$-graded algebra with Jacobson radical J_i. Moreover $A_i = B_i + J_i$, where B_i is a $G \times \mathbb{Z}_2$-graded simple algebra isomorphic to $F^{\alpha_i}H_i$ for some $H_i \leq G \times \mathbb{Z}_2$ and 2-cocycle $\alpha_i : H_i \times H_i \to F^*$.

Proof. By the previous lemma $\mathcal{V} = var^G(E(A))$ where $A = B^{(1)} \oplus \cdots \oplus B^{(m)} + J$ and, for every $i \in \{1, \ldots, m\}$, $B^{(i)} \cong F^{\alpha_i}H_i$ for some $H_i \leq G \times \mathbb{Z}_2$ and $\alpha_i : H_i \times H_i \to F^*$ a 2-cocycle.

Suppose that $B^{(i)}JB^{(k)} \neq 0$, for some $i \neq k$. Then there exist homogeneous elements $b_i \in B^{(i)}$, $b_k \in B^{(k)}$, $c \in J$ such that $b_icb_k \neq 0$. But $b_i = b_i1_{B^{(i)}}$, $b_j = b_j1_{B^{(j)}}$ implies $b_i1_{B^{(i)}}cb_j1_{B^{(j)}} \neq 0$. Set $f = 1_{B^{(i)}}$, $g = 1_{B^{(j)}}$, $h = 1_{B^{(i)}}c1_{B^{(j)}}$ and note that h is homogeneous and $f^2 = f$, $g^2 = g$, $fh = hg = h$, $hf = fg = gf = gh = 0$. Also f and g have homogeneous degree $(e, 0)$ and h has homogeneous degree (g, a), with $a = 0$ or 1. Thus if N is the algebra generated by f, g and h we have that $N \cong UT_2$ with $G \times \mathbb{Z}_2$ grading $\begin{pmatrix} (e,0) & (g,a) \\ 0 & (e,0) \end{pmatrix}$.

As we have seen in the proof of Lemma 3.1,

$$E(N) \cong \begin{pmatrix} E_0 & E_0 \\ 0 & E_0 \end{pmatrix} \quad \text{or} \quad \begin{pmatrix} E_0 & E_1 \\ 0 & E_0 \end{pmatrix}$$

with induced G-grading and it follows that $var(UT_2^G) = var(E(N)) \subseteq \mathcal{V}$, a contradiction.

Thus $B^{(i)}JB^{(k)} = 0$ for all $i \neq k$. Recall also that $B^{(i)}B^{(k)} = 0$ for all $i \neq k$. Clearly, these relations imply that the same relations hold for the Grassmann envelopes

$$E(B^{(i)})E(J)E(B^{(j)}) = 0, \ E(B^{(i)})E(B^{(j)}) = 0,$$

for all $i \neq j$. Set $A_i = B^{(i)} + J$, for $1 \leq i \leq m$. Then

$$A = B^{(1)} \oplus \cdots \oplus B^{(m)} + J = (B^{(1)} + J) + \cdots + (B^{(m)} + J) = A_1 + \cdots + A_m,$$

where $B^{(i)} + J = A_i$, $1 \leq i \leq m$.

We claim that $Id^G(E(A_1) \oplus \cdots \oplus E(A_m)) = Id^G(E(A_1)) \cap \cdots \cap Id^G(E(A_m))$.

In fact, if $f = f(x_1, \ldots, x_n) \in Id^G(E(A_1)) \cap \cdots \cap Id^G(E(A_m))$ is multilinear, we shall prove that $f \equiv 0$ on $E(A_1) + \cdots + E(A_m)$. To this end it suffices to check evaluations such that $\varphi(x_{i,g}) = \bar{x}_{i,g} \in E(A_1) \cup \cdots \cup E(A_m)$. Now if $\bar{x}_{1,g_1}, \ldots, \bar{x}_{n,g_n} \in E(A_j)$ for some j, then $f(\bar{x}_{1,g_1}, \ldots, \bar{x}_{n,g_n}) = 0$. If, say, $\bar{x}_{1,g_1} \in E(A_i)$ and $\bar{x}_{2,g_2} \in E(A_j)$ with $i \neq j$ then $\bar{x}_{\sigma(1),g_{\sigma(1)}} \cdots \bar{x}_{\sigma(n),g_{\sigma(n)}} = 0$, for all $\sigma \in S_n$, by the previous relations. Thus $f \in Id^G(E(A_1) \oplus \cdots \oplus E(A_m))$. Since the other inclusion is obvious we get the equality. It follows that $var^G(E(B)) = var^G(E(A_1) \oplus \cdots \oplus E(A_m))$. $\square$

4. On the $(n_1, \ldots, n_s)$th cocharacter

In this section, we study the $(n_1, \ldots, n_s)$th cocharacter of $\mathcal{V}$ and as a consequence we prove the main theorem.

We start by recalling some notation. If $d \geq 1$, l, t are integers, we define a partition whose diagram is hook shaped of arm d and leg l, as

$$h(d, l, t) = (\underbrace{l+t, \ldots, l+t}_{d}, \underbrace{l, \ldots, l}_{t}),$$

and then we set

$$H(d, l) = \bigcup_{n \geq 1} \{\lambda = (\lambda_1, \ldots, \lambda_r) \vdash n \mid \lambda_{d+1} \leq l\}.$$

Moreover, for any integer $a \geq 1$ we define

$$H(d, l) \cup (a^a) = \bigcup_{n \geq 1} \{\lambda = (\lambda_1, \ldots, \lambda_r) \vdash n \mid \lambda_{d+1} \leq l + a, \lambda_{d+a+1} \leq l\}.$$

We recall that if A is a $G \times \mathbb{Z}_2$-graded algebra and we consider the G-graded structure of A, then we write $A = \oplus_{j=1}^s A_{g_j}$ where $A_{g_j} = A_{(g_j,0)} \oplus A_{(g_j,1)}$.

Next we recall the following result that was proved in [9, Lemma 6].

Lemma 4.1. *Let $A = B + J$ be a finite dimensional $G \times \mathbb{Z}_2$-graded algebra,* $\dim A = m$*, with B a maximal $G \times \mathbb{Z}_2$-graded semisimple subalgebra. Let $\langle\lambda\rangle = (\lambda(1), \ldots \lambda(s)) \vdash n$ be a multipartition of n such that for some j, $1 \leq j \leq s$,*

$$\lambda(j) \geq h(d, p_j - d, (m+1)^2),$$

where d is an integer and $p_j = dimB_{g_j}$. Then $m_{\langle\lambda\rangle} = 0$ in the $(n_1, \ldots, n_s)$th cocharacter $\chi^G_{n_1,\ldots,n_s}(E(A))$ of $E(A)$.

The following result can be essentially found in [9, Lemma 7]. Here we give the proof for completeness.

Lemma 4.2. *Let $A = B + J$ be a finite dimensional $G \times \mathbb{Z}_2$-graded algebra with B a maximal semisimple graded subalgebra. Let $p_j = \dim B_{g_j}$, $1 \le j \le s$, and $m = \dim A$. If $\chi^G_{n_1,\dots,n_s}(E(A)) = \sum_{\langle\lambda\rangle\vdash n} m_{\langle\lambda\rangle}\chi_{\lambda(1)} \otimes \cdots \otimes \chi_{\lambda(s)}$, then $m_{\langle\lambda\rangle} \neq 0$ implies that $\langle\lambda\rangle = (\lambda(1),\dots,\lambda(s))$ where for $1 \le j \le s$, $\lambda(j) \subseteq H(d_j, p_j - d_j) \cup (u^u)$, for suitable integers $0 < d_j \le p_j$, with $u = (m+1)^2 + m$.*

Proof. Let $\langle\lambda\rangle \vdash n$ and suppose that $m_{\langle\lambda\rangle} \neq 0$. Write $\lambda(j) = (\lambda(j)_1, \lambda(j)_2, \dots)$, $1 \le j \le s$, and suppose that for some j there exists i such that $\lambda(j)_i > (m+1)^2 + m$. Let k be the integer such that $\lambda(j)_k > (m+1)^2 + m$ and $\lambda(j)_{k+1} \le (m+1)^2 + m$. If $k > p_j$ then $\lambda(j) \ge h(p_j + 1, 0, (m+1)^2)$ and we reach a contradiction by the previous lemma. Thus $k \le p_j$.

Set $u = (m+1)^2 + m$. If $\lambda(j)_{u+1} \ge p_j - k + 1$, then $\lambda(j) \ge \mu$ where

$$\mu = (\mu_1, \dots \mu_{u+1}) = ((u+1)^k, (p_j - k + 1)^{u+1-k}).$$

Since $(m+1)^2+m+1-(p_j-k+1) \ge (m+1)^2$ and $(m+1)^2+m+1-k \ge (m+1)^2$ we see that $\mu \ge h(k, p_j-k+1, (m+1)^2)$; hence $\lambda(j) \ge h(k, p_j - k + 1, (m+1)^2)$, again a contradiction by the previous lemma.

Thus $\lambda(j)_{u+1} \le p_j - k$ and $\lambda(j) \subseteq H(k, p_j - k) \cup (u^u)$. Therefore, we may assume that $\lambda(j)_1 \le (m+1)^2 + m$. Clearly, $\lambda(j)_{u+1} \le p_j$ since otherwise $\lambda(j) \ge h(0, p_j+1, (m+1)^2)$ contrary to the previous lemma. This says that $\lambda(j) \subseteq H(0, p_j) \cup (u^u)$ and we are done. □

As an immediate consequence of the previous lemma we get the following corollary.

Corollary 4.3. *Let $A = B + J$ be defined as in the previous lemma, $m = \dim A$, $p_j = \dim B_{g_j}$. If in $\chi_{n_1,\dots,n_s}(E(A)) = \sum_{\langle\lambda\rangle\vdash n} m_{\langle\lambda\rangle}\chi_{\lambda(1)} \otimes \cdots \otimes \chi_{\lambda(s)}$, $m_{\langle\lambda\rangle} \neq 0$, then for every $j \in \{1,\dots,s\}$, $\lambda(j) \in H(r_j, r_j)$, where $r_j = (m+1)^2 + m + p_j$.*

Lemma 4.4. *Let $A = B + J$ be a $G \times \mathbb{Z}_2$-graded algebra with J the Jacobson radical of A and $B \cong F^\alpha H$ for some $H \le G \times \mathbb{Z}_2$ and $\alpha : H \times H \to F^*$ a 2-cocycle. Then there exists a constant M such that $\chi^G_{n_1,\dots,n_s}(E(A)) = \sum_{\langle\lambda\rangle\vdash n} m_{\langle\lambda\rangle}\chi_{\lambda(1)} \otimes \cdots \otimes \chi_{\lambda(s)}$ and $m_{\langle\lambda\rangle} \le M$, for all $\langle\lambda\rangle \vdash n$ and for all $n \ge 1$.*

Proof. As we have seen in Corollary 4.3, the G-graded cocharacter of $E(A)$ lies in a s-tuple of hooks $H(r_j, r_j)$, $1 \le j \le s$. Choose a basis of A of $G\times\mathbb{Z}_2$-homogeneous elements. Let $m_j = \dim A_{(g_j,0)}$, $\overline{m}_j = \dim A_{(g_j,1)}$, $1 \le j \le s$. We have $A_{(g_j,k)} = B_{(g_j,k)} + J_{(g_j,k)}$ where $k = 0$ or $k = 1$, and since $B \cong F^\alpha H$, we have that $\dim B_{(g_j,k)} \le 1$. So let

$$\begin{array}{ll} A_{(g_1,0)} = \mathrm{span}\{a^1_0, \dots, a^1_{m_1-1}\} & A_{(g_1,1)} = \mathrm{span}\{b^1_0, \dots, b^1_{\overline{m}_1-1}\} \\ \vdots & \vdots \\ A_{(g_s,0)} = \mathrm{span}\{a^s_0, \dots, a^s_{m_s-1}\} & A_{(g_s,1)} = \mathrm{span}\{b^s_0, \dots, b^s_{\overline{m}_s-1}\}, \end{array}$$

where $a^j_0 \in B_{(g_j,0)}$ if $B_{(g_j,0)} \neq 0$, and similarly $b^j_0 \in B_{(g_j,1)}$ if $B_{(g_j,1)} \neq 0$, $1 \le j \le s$. All other elements a^j_i, b^j_l lie in J, $1 \le i \le m_j - 1$, $1 \le l \le \overline{m}_j - 1$.

Let q be the least positive integer such that $J^q = 0$ and set

$$N_0 = ((2q)^{\sum_{j=1}^s m_j + \overline{m}_j})^{2sr_j}.$$

We shall prove that every $m_{\langle\lambda\rangle}$ in $\chi^G_{n_1,\dots,n_s}(E(A))$ is bounded by $M = (\sum_{j=1}^s m_j + \overline{m}_j)N_0$.

To this end let $\langle\lambda\rangle = (\lambda(1),\dots,\lambda(s))$ be such that $\lambda(j) \in H(r_j, r_j)$, $1 \le j \le s$, and consider $T_{\langle\lambda\rangle} = (T_{\lambda(1)},\dots,T_{\lambda(s)})$, a corresponding multitableau. For every tableau $T_{\lambda(j)}$ let $R_{\lambda(j)}$ and $C_{\lambda(j)}$ be the row stabilizer and the column stabilizer of $T_{\lambda(j)}$, respectively. Let $R^+_{\lambda(j)} = \sum_{\sigma\in R_{\lambda(j)}} \sigma$, $C^-_{\lambda(j)} = \sum_{\tau\in C_{\lambda(j)}} (\mathrm{sgn}\tau)\tau$ and let $e_{T_{\lambda(j)}} = R^+_{\lambda(j)} C^-_{\lambda(j)}$ denote the corresponding essential idempotent of the group algebra FS_{n_j}. Then $e_{T_{\langle\lambda\rangle}} = e_{T_{\lambda(1)}} \cdots e_{T_{\lambda(s)}}$ is an essential idempotent of $F(S_{n_1} \times \cdots \times S_{n_s})$.

For every $j = 1,\dots,s$, consider the group $K_j = \{\sigma \in C_{\lambda(j)} \mid \sigma(i) = i$, for every i out of the first r_j columns$\}$, and let $K^-_j = \sum_{\sigma\in K_j} (-1)^\sigma \sigma$. Then define $K^- = K^-_1 \cdots K^-_s$ and notice that, since each $e_{T_{\lambda(j)}}$ is an essential idempotent, then $K^- e_{T_{\langle\lambda\rangle}} \neq 0$ and $e_{T_{\langle\lambda\rangle}}$ generate the same minimal left ideal of $F(S_{n_1} \times \cdots \times S_{n_s})$.

For every j, let Y^j_i be the set of variables of homogeneous degree g_j, whose indices lie in the i-th column of $\lambda(j)$. Let also X^j_i be the set of variables of homogeneous degree g_j whose indices lie in the ith row of $\lambda(j)$ but do not belong to the first r_j columns. Then, for every polynomial $f \in P_{n_1,\dots,n_s}$, $K^-_j e_{T_{\lambda(j)}} f$ is alternating on each of the sets $Y^j_1,\dots,Y^j_{r_j}$ and is symmetric on each of the sets $X^j_1,\dots,X^j_{r_j}$. Thus, if we now consider the polynomial $g = K^- e_{T_{\langle\lambda\rangle}} f$, the variables of g are partitioned into $2r_1 + \cdots + 2r_s$ disjoint subsets

$$X^1_1,\dots,X^1_{r_1}, Y^1_1,\dots,Y^1_{r_1},\dots,X^s_1,\dots,X^s_{r_s}, Y^s_1,\dots,Y^s_{r_s}$$

and g is symmetric or alternating on each set as described above.

For every $j = 1,\dots,s$ and $i = 1,\dots,r_j$ if $\lambda(j) = (\lambda(j)_1, \lambda(j)_2, \dots)$, then X^j_i is empty if $\lambda(j)_i \le r_j$, i.e, if the length of the i-th row of $T_{\lambda(j)}$ is less than or equal to r_j. On the other hand, if $\lambda(j)_i > r_j$ then $|X^j_i| = \lambda(j)_i - r_j$. Moreover, $|Y^j_i| = \lambda(j)'_i$ where $\lambda(j)' = (\lambda(j)'_1, \lambda(j)'_2, \dots)$ is the conjugate partition of $\lambda(j)$.

Notice that for any $\rho_j \in S_{n_j}$ we also have $\rho_j K_j^- e_{T_{\lambda(j)}} \neq 0$, and so if $\rho = \rho_1 \cdots \rho_s \in S_{n_1} \times \cdots \times S_{n_s}$, $\rho K^- e_{T_{\langle\lambda\rangle}} = \rho_1 K_1^- e_{T_{\lambda(1)}} \cdots \rho_s K_s^- e_{T_{\lambda(s)}} \neq 0$. It follows that if $f \in P_{n_1,\ldots,n_s}$ is such that $e_{T_{\langle\lambda\rangle}} f \neq 0$, then the polynomials $e_{T_{\langle\lambda\rangle}} f$ and $g' = \rho K^- e_{T_{\langle\lambda\rangle}} f$ generate the same irreducible $S_{n_1} \times \cdots \times S_{n_s}$-module. Now we choose ρ_j, $1 \le j \le s$, in such a way that $\rho_j K_j^- e_{T_{\lambda(j)}} f$ is symmetric separately on the first $\lambda(j)_1 - r_j$ variables, on the next $\lambda(j)_2 - r_j$ variables and so on. A similar condition holds for the alternating sets of variables Y_i^j, $1 \le i \le r_j$. The corresponding property of the polynomial g' is clear.

Let now $f_1, \ldots, f_M \in P_{n_1,\ldots,n_s}$ be multilinear polynomials such that $F(S_{n_1} \times \cdots \times S_{n_s})f_i \cong F(S_{n_1} \times \cdots \times S_{n_s})f_j$, for all $i, j = 1, \ldots, M$, i.e., $f_1, \ldots, f_M$ generate irreducible $S_{n_1} \times \cdots \times S_{n_s}$-modules corresponding to the same multipartition $\langle\lambda\rangle$. By what we remarked above, we can choose permutations $\rho_1, \ldots, \rho_M \in S_{n_1} \times \cdots \times S_{n_s}$ and a decomposition $X^1 \cup \cdots \cup X^s \cup Y^1 \cup \cdots \cup Y^s$, where for every $j = 1, \ldots, s$, $X^j = X_1^j \cup \cdots \cup X_{r_j}^j$, $Y^j = Y_1^j \cup \cdots \cup Y_{r_j}^j$ are sets of variables of homogeneous degree g_j and $\rho_1 f_1, \ldots, \rho_M f_M$ are simultaneously symmetric on X_i^j and alternating on Y_i^j, for all $j = 1, \ldots, s$, $i = 1, \ldots, r_j$.

Assume by contradiction that $m_{\langle\lambda\rangle} = M \ge \sum_{j=1}^s (m_j + \overline{m}_j) N_0$. We shall prove that $E(A)$ satisfies an identity of the type

$$f = \gamma_1 f_1 + \cdots + \gamma_M f_M \equiv 0, \tag{3}$$

where $\gamma_1, \ldots, \gamma_M \in F$ are not all zero. Clearly, it is sufficient to verify that f has only zero values on elements of the form $a_k^j \otimes e$ and $b_l^j \otimes e'$, where $a_k^j \in A_{(g_j,0)}$, $e \in E_0$, $b_l^j \in A_{(g_j,1)}$, $e' \in E_1$ and $k \in \{0, \ldots, m_j - 1\}$, $l \in \{0, \ldots \overline{m}_j - 1\}$.

First we define special substitutions as follows. Let

$$0 \le \alpha_0^{ji}, \alpha_1^{ji}, \ldots, \alpha_{m_j-1}^{ji}, \beta_0^{ji}, \beta_1^{ji}, \ldots, \beta_{\overline{m}_j-1}^{ji}$$

be integers satisfying the following equalities:

$$\sum_{k=0}^{m_j-1} \alpha_k^{ji} + \sum_{k=0}^{\overline{m}_j-1} \beta_k^{ji} = |X_i^j|$$
$$\sum_{k=0}^{m_j-1} \alpha_k^{j(r_j+i)} + \sum_{k=0}^{\overline{m}_j-1} \beta_k^{j(r_j+i)} = |Y_i^j|$$
$$1 \le j \le s,\ 1 \le i \le r_j.$$

We say that a substitution φ has type $0 \le \alpha_0^{ji}, \alpha_1^{ji}, \ldots, \alpha_{m_j-1}^{ji}, \beta_0^{ji}, \beta_1^{ji}, \ldots, \beta_{\overline{m}_j-1}^{ji}$, $1 \le j \le s$, $1 \le i \le r_j$, if we replace the variables in the following way: for fixed i and j, we replace the first α_0^{ji} variables from X_i^j by elements $a_0^j \otimes e$ (with distinct elements e for distinct $x \in X_i^j$), the next α_1^{ji} variables by elements $a_1^j \otimes e$ and so on, where all elements e lie in E_0. Now substitute the following β_0^{ji} variables from X_i^j by elements $b_0^j \otimes e'$, the next by $b_1^j \otimes e'$, and so on where all elements e' lie in E_1. We apply the same procedure in order to replace the variables in Y_i^j by elements of the type $a_k^j \otimes e$ and $b_k^j \otimes e'$.

In order to obtain a non zero value of the polynomials in (3), any substitution above should satisfy the following restrictions.

(1) $\beta_k^{ji} \le 1, \quad 1 \le j \le s, 1 \le i \le r_j$, where $0 \le k \le \overline{m}_j - 1$.
(2) $\alpha_1^{ji} + \cdots + \alpha_{m_j-1}^{ji} \le q - 1, \quad 1 \le j \le s, 1 \le i \le r_j$.
(3) $\alpha_0^{ji} = |X_i^j| - (\alpha_1^{ji} + \cdots + \alpha_{m_j-1}^{ji} + \beta_0^{ji} + \cdots + \beta_{\overline{m}_j-1}^{ji})$.

The first property follows since f_j is symmetric on X_i^j and, so, it becomes zero when we evaluate two variables of X_i^j in $b_u^j \otimes e'$, $b_u^j \otimes e''$, for some $e', e'' \in E_1$. The second property follows since $J^q = 0$.

Similarly, we replace the variables from Y_i^j by elements of the form $a_u^j \otimes e$, $b_u^j \otimes e'$ as above and we obtain the following restrictions of the integers $\alpha_k^{j(r_j+i)}, \beta_k^{j(r_j+i)}$, for $1 \le j \le s$, $1 \le i \le r_j$ and $0 \le k \le m_j - 1$.

(1) $\alpha_k^{ji} \le 1, \quad 1 \le j \le s, r_j + 1 \le i \le 2r_j$ where $0 \le k \le m_j - 1$.
(2) $\beta_1^{ji} + \cdots + \beta_{\overline{m}_j-1}^{ji} \le q - 1, \quad 1 \le j \le s, r_j + 1 \le i \le 2r_j$.
(3) $\beta_0^{ji} = |Y_i^j| - (\beta_1^{ji} + \cdots + \beta_{\overline{m}_j-1}^{ji} + \alpha_0^{ji} + \cdots + \alpha_{m_j-1}^{ji})$.

Now, from the restrictions 1, 2, 3 above we get that for each $j = 1, \ldots, s$, $i = 1, \ldots, r_j$, the number of distinct $\overline{m}_j$-tuples $(\beta_0^{ji}, \ldots, \beta_{\overline{m}_j-1}^{ji})$ is at most $2^{\overline{m}_j}$ and the number of distinct m_j-tuples $(\alpha_0^{ji}, \ldots, \alpha_{m_j-1}^{ji})$ is at most q^{m_j}. Thus the number of distinct $m_j + \overline{m}_j$-tuples $(\alpha_0^{ji}, \ldots, \beta_{\overline{m}_j-1}^{ji})$ is at most $2^{\overline{m}_j} q^{m_j} < (2q)^{m_j + \overline{m}_j}$. Similarly, from the other three conditions, we get that the number of distinct $m_j + \overline{m}_j$-tuples $(\alpha_0^{j(r_j+i)}, \ldots, \beta_{\overline{m}_j-1}^{j(r_j+i)})$ is bounded by $(2q)^{m_j + \overline{m}_j}$. It follows that the total number N of distinct types of substitutions is less than $((2q)^{\sum_{j=1}^s m_j + \overline{m}_j})^{\sum_{j=1}^s 2r_j} = N_0$.

Notice that if φ, φ' are two substitutions of the same type and $\varphi(z) = u \otimes p$ for some $z \in X$, $u \in A$, $p \in E$, then $\varphi'(z) = u \otimes p'$ with the same grading of the elements p, p'. Hence if $X = \{z_1, \ldots, z_n\}$, $\varphi(z_i) = u_i \otimes p_i$ and $\varphi'(z_i) = u_i \otimes p'_i$, then

$$\varphi(f) = f(u_1 \otimes p_1, \ldots, u_n \otimes p_n) = w \otimes p_1 \cdots p_n$$
$$\varphi'(f) = f(u_1 \otimes p'_1, \ldots, u_n \otimes p'_n) = w \otimes p'_1 \cdots p'_n.$$

In this case, we say that φ and φ' are similar. Let N be the number of similarity classes.

Now let $\varphi_1, \ldots, \varphi_N$ be substitutions, chosen one from each similarity class of distinct types. If φ is one of these substitutions, and h_1, h_2 are two multilinear polynomials of degree n, then by multilinearity and supercommutativity $\varphi(h_1) = r_1 \otimes p_1 \cdots p_n$ and $\varphi(h_2) = r_2 \otimes p_1 \cdots p_n$, where $p_1, \ldots, p_n \in E$ and $r_1, r_2 \in A$. Therefore, for each $j = 1, \ldots, N$ and $i = 1, \ldots, M$ we get

$$\varphi_j(f_i) = a_{ij} \otimes p_{j1} \cdots p_{jn},$$

where $a_{ij} \in A$ and $p_{j1}, \ldots, p_{jn}$ depend on φ_j only.

We consider the matrix (a_{ij}), $1 \le i \le M$, $1 \le j \le N$, whose elements a_{ij} lie in A. Since $M = (\sum_{k=1}^{s}(m_k + \overline{m}_k))N_0$, where $\dim A = \sum_{k=1}^{s}(m_k + \overline{m}_k)$, the rows of (a_{ij}) are linearly dependent. Hence there exist $\gamma_1, \ldots, \gamma_M \in F$ not all zero, such that

$$\sum_{i=1}^{M} \gamma_i a_{ij} = 0, \quad 1 \le j \le N.$$

From the above we get $\varphi_j(\sum_{i=1}^{M} \gamma_i f_i) = \sum_{i=1}^{M} \gamma_i \varphi_j(f_i) = (\sum_{i=1}^{M} \gamma_i a_{ij}) \otimes p_{j1} \cdots p_{jn} = 0$, for all $1 \le j \le N$.

We claim that this implies that $f = \sum_{i=1}^{M} \gamma_i f_i$ is an identity of $E(A)$. In fact by multilinearity it is enough to check only substitutions φ^* where the variables are evaluated into elements of the type $u \otimes p$, where $u = a_i^j$ or b_i^j, for some i, j and $p \in E_0 \cup E_1$.

Now, there exists a permutation σ of the variables (preserving the homogeneous degree) such that $\varphi^*\sigma = \varphi'$ is similar to some φ_j, $1 \le j \le N$. Thus $\varphi'(f_i) = a_{ij} \otimes p'_{j1} \cdots p'_{jn}$ and, so, $\varphi'(f) = 0$. We remark that the above σ satisfies $\sigma(X_i^j) = X_i^j$ and $\sigma(Y_i^j) = Y_i^j$ $1 \le j \le s$, $1 \le i \le r_j$. Since f is symmetric on X_i^j and alternating on Y_i^j, it follows that $\varphi'(f) = \varphi^*\sigma(f) = \varphi(\pm f) = \pm\varphi^*(f) = 0$. Thus $\varphi^*(f) = 0$. We have shown that modulo the identities of $E(A)$, any M polynomials corresponding to the same multitableau are linearly dependent and this is equivalent to say that $m_{\langle\lambda\rangle} \le M$ for all $\langle\lambda\rangle \vdash n$. This completes the proof of the lemma. $\square$

Lemma 4.5. *Let $A = B + J$, $B \cong F^\alpha H$ for some H subgroup of $G \times \mathbb{Z}_2$ and $\alpha : H \times H \to F^*$ a 2-cocycle. Then there exists a constant M such that $\chi^G_{n_1,\ldots,n_s}(E(A)) = \sum_{\langle\lambda\rangle\vdash n} m_{\langle\lambda\rangle}\chi_{\lambda(1)} \otimes \cdots \otimes \chi_{\lambda(s)}$ and*

$$n_j - \lambda(j)_1 - \lambda(j)'_1 \le M, \quad 1 \le j \le s.$$

Proof. Let $\langle\lambda\rangle = (\lambda(1), \ldots, \lambda(s)) \vdash n$ be a multipartition of n and let q be such that $J^q = 0$. We claim that if $m_{\langle\lambda\rangle} \ne 0$, then $\lambda(j)_2 \le q + 1$, for all j, $1 \le j \le s$, that is the diagram of each $\lambda(j)$ contains at most $q + 1$ boxes in the second row.

In fact, suppose by contradiction that there exists j, $1 \le j \le s$, such that $\lambda(j)_2 \ge q + 2$ and $m_{\langle\lambda\rangle} \ne 0$. Then there exists a multitableau $T_{\langle\lambda\rangle} = (T_{\lambda(1)}, \ldots, T_{\lambda(s)})$, a corresponding essential idempotent $e_{T_{\langle\lambda\rangle}} = e_{T_{\lambda(1)}} \cdots e_{T_{\lambda(s)}}$ and a polynomial $f \in P_{n_1,\ldots,n_s}$ such that $e_{T_{\langle\lambda\rangle}} f \notin Id^G(E(A))$. Recall that $e_{T_{\lambda(j)}}$ acts on n_j variables of homogeneous degree $g_j \in G$. Since $e_{T_{\lambda(j)}}$ is an essential idempotent, there exists $\tau \in R_{\lambda(j)}$ such that $\tau C^-_{\lambda(j)} e_{T_{\langle\lambda\rangle}} f \notin Id^G(E(A))$. Let $i_1, \ldots, i_{q+2}$ denote the integers in the first $q+2$ positions of the first row of the diagram of $\lambda(j)$ written from left to right. Similarly, let $k_1, \ldots, k_{q+2}$ be the integers in the first $q+2$ positions of the second row of $\lambda(j)$. Then the polynomial $g = \tau C^-_{\lambda(j)} e_{T_{\langle\lambda\rangle}} f$ is alternating on each of the following sets: $\{x_{\tau(i_1),g_j}, x_{\tau(k_1),g_j}\}, \ldots, \{x_{\tau(i_{q+2}),g_j}, x_{\tau(k_{q+2}),g_j}\}$.

Notice that these variables are evaluated in

$$E(A)_{g_j} = ((E_0 \otimes B_{(g_j,0)}) \oplus (E_0 \otimes J_{(g_j,0)})) \oplus ((E_1 \otimes B_{(g_j,1)}) \oplus (E_1 \otimes J_{(g_j,1)}))$$

and, since $B \cong F^\alpha H$, the spaces $B_{(g_j,0)}$ and $B_{(g_j,1)}$ are at most 1-dimensional. Now, if at least q of the above variables are evaluated in $E_0 \otimes J_{(g_j,0)} \cup E_1 \otimes J_{(g_j,1)}$, then we get that g vanishes in $E(A)$ since $J^q = 0$.

Therefore, there exist three sets among

$$\{x_{\tau(i_1),g_j}, x_{\tau(k_1),g_j}\}, \ldots, \{x_{\tau(i_{q+2}),g_j}, x_{\tau(k_{q+2}),g_j}\}$$

that are evaluated in $(E_0 \otimes B_{(g_j,0)}) \cup (E_1 \otimes B_{(g_j,1)})$. If one of these sets, say $\{x_{\tau(i_1),g_j}, x_{\tau(k_1),g_j}\}$, is evaluated in the commutative algebra $(E_0 \otimes B_{(g_j,0)})$, then we will get $g \equiv 0$ on $E(A)$, since g is alternating in $x_{\tau(i_1),g_j}$ and $x_{\tau(k_1),g_j}$. Then we deduce that there are at least two variables corresponding to indices in the same first row or second row of $T_{\lambda(j)}$, say $x_{\tau(i_1),g_j}$ and $x_{\tau(i_2),g_j}$ that are evaluated in $E_1 \otimes B_{(g_j,1)}$.

Now the polynomial $e_{T_{\langle\lambda\rangle}} f$ is symmetric on the set $\{x_{i_1,g_j}, \ldots, x_{i_{q+2},g_j}\}$; hence, since $\tau \in R_{\lambda(j)}$, it is also symmetric on $\{x_{\tau(i_1),g_j}, \ldots, x_{\tau(i_{q+2}),g_j}\}$. Since the variables $x_{\tau(i_1),g_j}$ and $x_{\tau(i_2),g_j}$ are evaluated in $E_1 \otimes B_{(g_j,1)}$, which is anticommutative, we get that $e_{T_{\langle\lambda\rangle}} f \equiv 0$ on $E(A)$ and the claim is proved.

Next we claim that if $m_{\langle\lambda\rangle} \neq 0$ then $\lambda(j)'_2 \leq 2q$, for all j, $1 \leq j \leq s$. This is the same as to say that the diagram of each $\lambda(j)$ contains at most $2q$ boxes in the second column.
In fact, suppose to the contrary that there exists j, $1 \leq j \leq s$, such that $\lambda(j)'_2 \geq 2q+1$ and $m_{\langle\lambda\rangle} \neq 0$. As above, this says that there exists a multitableau $T_{\langle\lambda\rangle} = (T_{\lambda(1)}, \ldots, T_{\lambda(s)})$, an essential idempotent $e_{T_{\langle\lambda\rangle}} = e_{T_{\lambda(1)}} \cdots e_{T_{\lambda(s)}}$ and a polynomial $f \in P_{n_1,\ldots,n_s}$ such that $e_{T_{\langle\lambda\rangle}} f \notin Id^G(E(A))$. Let $\tau \in R_{\lambda(j)}$ be such that $g = \tau C_{\lambda(j)} e_{T_{\langle\lambda\rangle}} f \notin Id^G(E(A))$. Let $i_1, \ldots, i_{2q+1}$ be the first integers in the first column of $T_{\lambda(j)}$ written from top to bottom and $k_1, \ldots, k_{2q+1}$ the corresponding integers of the second column. Then g is alternating on $\{x_{\tau(i_1),g_j}, \ldots, x_{\tau(i_{2q+1}),g_j}\}$ and on $\{x_{\tau(k_1),g_j}, \ldots, x_{\tau(k_{2q+1}),g_j}\}$.

In order to get a non zero value of g, since $E_0 \otimes B_{(g_j,0)}$ is commutative, we can evaluate at most one variable of each set in $E_0 \otimes B_{(g_j,0)}$. Moreover since $J^q = 0$, we have to evaluate at most $q-1$ variables of each set into $E_1 \otimes B_{(g_j,1)}$. It follows that two variables corresponding to indices in the same row, say $x_{\tau(i_1),g_j}$ and $x_{\tau(k_1),g_j}$, are evaluated into $E_1 \otimes B_{(g_j,1)}$. Since g is symmetric on these two variables and $E_1 \otimes B_{(g_j,1)}$ is anticommutative, we get $g \equiv 0$, a contradiction. This proves the second claim.

As a result of the above two claims we get that if $m_{\langle\lambda\rangle} \neq 0$, then $\lambda(j)_2 \leq q+1$ and $\lambda(j)'_2 \leq 2q$. As a result, the diagram of $\lambda(j)$ out of the first row and first column is contained in a $q \times (2q-1)$ rectangle. In other words, $n_j - \lambda(j)_1 - \lambda(j)'_1 \leq q(2q-1)$, for all $1 \leq j \leq s$. Thus $M = q(2q-1)$ is the desired constant, and the proof is complete. □

We are now ready to prove our main theorem.

Theorem 4.6. *Let A be a G-graded PI-algebra, and*

$$\chi^G_{n_1,\ldots,n_s}(A) = \sum_{\langle\lambda\rangle \vdash n} m_{\langle\lambda\rangle} \chi_{\lambda(1)} \otimes \cdots \otimes \chi_{\lambda(s)}$$

its $(n_1, \ldots, n_s)$th cocharacter. Then the following conditions are equivalent.

(1) *There exists a constant M such that for all n and $\langle\lambda\rangle \vdash n$, the inequality*

$$m_{\langle\lambda\rangle} \leq M$$

holds.

(2) *$Id^G(A) \not\subseteq Id^G(UT_2^G)$, for any G-grading on UT_2.*

(3) *There exists a constant N such that for all n and $\langle\lambda\rangle \vdash n$, the inequalities*

$$n_i - \lambda(i)_1 - \lambda(i)'_1 \leq N$$

hold, for all i, $1 \leq i \leq s$.

Proof. Let $\mathcal{V} = \mathrm{var}^G(A)$. If $UT_2^G \in \mathcal{V}$ for some G-grading on UT_2, then by Theorem 2.3 the multiplicities in $\chi^G_{n_1,\ldots,n_s}(UT_2^G)$, and so, in $\chi^G_{n_1,\ldots,n_s}(\mathcal{V})$ are not bounded by a constant. This proves (1) ⇒ (2).

Suppose that $UT_2^G \notin \mathcal{V}$, for any G-grading on UT_2. By Lemma 3.2, we can write $\mathcal{V} = \mathrm{var}^G(E(A_1) \oplus \cdots \oplus E(A_n))$ where for every i, $1 \leq i \leq n$, $A_i = B_i + J_i$, with B_i a $G \times \mathbb{Z}_2$-graded simple algebra isomorphic to $F^{\alpha_i} H_i$ for some $H_i \leq G \times \mathbb{Z}_2$ and $\alpha_i : H_i \times H_i \to F^*$ a 2-cocycle.

Now let $\chi^G_{n_1,\ldots,n_s}(E(A_i)) = \sum_{\langle\lambda\rangle \vdash n} m^{(i)}_{\langle\lambda\rangle} \chi_{\lambda(1)} \otimes \cdots \otimes \chi_{\lambda(s)}$, $1 \leq i \leq s$. Then

$$\sum_{\langle\lambda\rangle \vdash n} m_{\langle\lambda\rangle} \chi_{\lambda(1)} \otimes \cdots \otimes \chi_{\lambda(s)} = \chi^G_{n_1,\ldots,n_s}(\mathcal{V}) = \chi^G_{n_1,\ldots,n_s}(E(A_1) \oplus \cdots \oplus E(A_s)) \leq \sum_{\langle\lambda\rangle \vdash n} \left(\sum_{i=1}^{s} m^{(i)}_{\langle\lambda\rangle} \right) \chi_{\lambda(1)} \otimes \cdots \otimes \chi_{\lambda(s)}.$$

Since by Lemma 4.4, $m^{(i)}_{\langle\lambda\rangle} \leq M_i$, for some constant M_i, we get that $m_{\langle\lambda\rangle} \leq \sum_{i=1}^{s} M_i$ is bounded by a constant, for all $\langle\lambda\rangle \vdash n$. This proves (2) ⇒ (1).

The implication (2) ⇒ (3) was proved in Lemma 4.5.

We next prove that (3) ⇒ (2). Suppose by contradiction that $UT_2^G \in \mathcal{V}$ for some G-grading on UT_2. If

$$\chi^G_{n_1,\ldots,n_s}(UT_2^G) = \sum_{\langle\lambda\rangle \vdash n} m'_{\langle\lambda\rangle} \chi_{\lambda(1)} \otimes \cdots \otimes \chi_{\lambda(s)}$$

and

$$\chi^G_{n_1,\ldots,n_s}(\mathcal{V}) = \sum_{\langle\lambda\rangle \vdash n} m_{\langle\lambda\rangle} \chi_{\lambda(1)} \otimes \cdots \otimes \chi_{\lambda(s)},$$

then since $UT_2^G \in \mathcal{V}$, we get that $m'_{\langle\lambda\rangle} \leq m_{\langle\lambda\rangle}$, for all $\langle\lambda\rangle \vdash n$. So, for every $\langle\lambda\rangle \vdash n$ such that $m'_{\langle\lambda\rangle} \neq 0$, we have that $n_i - \lambda(i)_1 - \lambda(i)'_1 \leq N$, for some constant N and for all i, $1 \leq i \leq s$.

Now take $n = 2N+5$ and $\langle\lambda\rangle = ((N+2, N+2), (1), \emptyset, \ldots, \emptyset) \vdash n$; hence $\lambda(1) = (N+2, N+2) \vdash 2N+4$, $\lambda(2) = (1)$ and $\lambda(i) = \emptyset$ for all $i \geq 3$. Then, according to Theorem 2.3, $m'_{\langle\lambda\rangle} = 1 \neq 0$, but $2N+4-(N+1)-2 = N+1 > N$. Thus $m_{\langle\lambda\rangle} \geq m'_{\langle\lambda\rangle} > N$, a contradiction. □

References

[1] E. Aljadeff, A. Giambruno, D. La Mattina, Graded polynomial identities and exponential growth, J. Reine Angew. Math. 650 (2011) 83–100.
[2] E. Aljadeff, A. Giambruno, Multialternating graded polynomials and growth of polynomial identities, Proc. Amer. Math. Soc. (in press).
[3] E. Aljadeff, A. Kanel-Belov, Representability and Specht problem for G-graded algebras, Adv. Math. 225 (2010) 2391–2428.
[4] Y.A. Bahturin, V. Drensky, Graded polynomial identities of matrices, Linear Algebra Appl. 357 (2002) 15–34.
[5] Y.A. Bahturin, M.V. Zaicev, S.K. Sehgal, Finite-dimensional simple graded algebras, Sb. Math. 199 (2008) 965–983.
[6] C.W. Curtis, I. Reiner, Representation Theory of Finite Groups and Associative Algebras, Wiley Classics Lib., John Wiley & Sons, Inc., New York, 1988.
[7] O.M. Di Vincenzo, P. Koshlukov, A. Valenti, Gradings on the algebra of upper triangular matrices and their graded identities, J. Algebra 275 (2004) 550–566.
[8] V. Drensky, A. Giambruno, Cocharacters, codimensions and Hilbert series of the polynomial identities for 2×2 matrices with involution, Canadian J. Math. 46 (1994) 718–733.
[9] A. Giambruno, D. La Mattina, Graded polynomial identities and codimensions: Computing the exponential growth, Adv. Math. 225 (2010) 859–881.
[10] A. Giambruno, A. Regev, Wreath products and PI-algebras, J. Pure Appl. Algebra 35 (1985) 133–149.
[11] A. Giambruno, M. Zaicev, Exponential codimension growth of P.I. algebras: an exact estimate, Adv. Math. 142 (1999) 221–243.
[12] A. Giambruno, M. Zaicev, Polynomial Identities and Asymptotic Methods, in: Mathematical Surveys and Monographs, vol. 122, AMS, Providence, RI, 2005.
[13] G. James, A. Kerber, The Representation Theory of the Symmetric Group, in: Encyclopedia of Mathematics and its applications, vol. 16, Addison-Wesley, London, 1981.
[14] S.P. Mishchenko, A. Regev, M. Zaicev, A characterization of P.I. algebras with bounded multiplicities of the cocharacters, J. Algebra 219 (1999) 356–368.
[15] I. Sviridova, Identities of PI-algebras graded by a finite abelian group, Comm. Algebra 39 (2011) 3462–3490.
[16] E. Taft, Orthogonal conjugacies in associative and Lie algebras, Trans. Amer. Math. Soc. 113 (1964) 18–29.
[17] A. Valenti, The graded identities of upper triangular matrices of size two, J. Pure Appl. Algebra 172 (2002) 325–335.